

Time-of-arrival probabilities and quantum measurements

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Abstract

We study the construction of probability densities for time-of-arrival in quantum mechanics. Our treatment is based upon the facts that (i) time appears in quantum theory as an external parameter to the system, and (ii) propositions about the time-of-arrival appear naturally when one considers histories. The definition of time-of-arrival probabilities is straightforward in stochastic processes. The difficulties that arise in quantum theory are due to the fact that the time parameter of Schrödinger's equation does not naturally define a probability density at the continuum limit, but also because the procedure one follows is sensitive on the interpretation of the reduction procedure. We consider the issue in Copenhagen quantum mechanics and in history-based schemes like consistent histories. The benefit of the latter is that it allows a proper passage to the continuous limit—there are however problems related to the quantum Zeno effect and decoherence. We finally employ the histories-based description to construct Positive-Operator-Valued-Measures (POVMs) for the time-of-arrival, which are valid for a general Hamiltonian. These POVMs typically depend on the resolution of the measurement device; for a free particle, however, this dependence cancels in the physically relevant regime and the POVM coincides with that of Kijowski.

1 Introduction

The probabilities provided by quantum mechanics usually refer to measurements that take place at specific moments of time. However, there are physically real-

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isable measurements that do not fall in this category. One such example involves the determination of a particle's time-of-arrival (or time-of-flight) measurement. If the quantum mechanical system is described by a wave function $\psi(x, t)$, its modulus square $|\psi(x, t)|^2$ is a probability density with respect to x at any time t . If ψ describes a particle beam, the probability density describes the particles' distribution in space, as if a snapshot were taken at a moment t . However, the set-up for particle detection is slightly different. The time interval between the emission of the beam and its detection is not fixed; rather one places a detector at a fixed distance L from the source. This is *not* a single-time measurement of the particle's position. Therefore, the object $|\psi(x, t)|^2$ is not immediately relevant, because it is not a probability distribution with respect to t . Only if one assumes that the initial wave-packet is narrowly concentrated around a specific momentum value p and that it remains so at all times until detection, is it possible to state that this measurement is equivalent to a single-time measurement at time $t = mL/p$ (m being the particle mass). But even for small momentum spreads of the initial state, the assumptions above are valid only for free particles.

In the general case, the detector registers particles at different times t . One therefore needs to construct a probability $p(t)$ for the time-of-arrival. This would have been an elementary problem, if there existed an operator representing time in the system's Hilbert space. In that case one would use the Born rule to determine a probability density for the time-of-arrival. Unfortunately the existence of a time operator \hat{T} , conjugate to the Hamiltonian (so that the Hamiltonian generates time-translations: $e^{i\hat{H}s}\hat{T}e^{-i\hat{H}s} = \hat{T} + s$) is ruled out by the requirement that the Hamiltonian is bounded from below [1, 2]. Nonetheless, one may still define a quantum time variable, by choosing some degrees of freedom of the system (or of a measuring apparatus) as defining an internal "clock". The simplest example for the time-of-arrival is the quantum version of the classical function mx/p , for a free particle of mass m — x, p being the position and momentum respectively. Still, clock times fail to be conjugate to the Hamiltonian of the system, with the result that they do not forward under Hamiltonian evolution. In simpler language, quantum fluctuations invariably force clocks to move occasionally "backwards in time". For time-of-arrival operators see for example [3, 4] and the extensive bibliography in the review [5].

This discussion brings us invariably to the issue of the role of time in quantum theory. Time enters the quantum mechanical formalism as the *external* evolution parameter in Schrödinger's equation; it is not an intrinsic variable of a physical system. Indeed, outside the realm of general relativity, time is assumed to be part of a background structure—both in non-relativistic or special relativistic physics. For Bohr, time is a part of the classical description of physics that is "complementary" to the quantum description, and is needed if the measurement theory of the later is to make sense.

In this paper we identify different candidates for the time-of-arrival proba-

bility, by treating time as a parameters external to the quantum system under consideration. This assumption is not only forced on us by the quantum mechanical formalism, but also corresponds to the way time is taken into account in experiments. When an event happens (say a detector clicks) we record the time it took place by "looking" at a clock in the laboratory. Clocks are classical systems that do not interact with either the quantum system or with the measurement device— their degrees of freedom are not correlated to the physical processes that take place in the act of measurement. In effect, the time parameter we consider is the reading of the clock that is simultaneous with the detector's click. Since in any laboratory the relative speeds involved are much smaller than the speed of light, there is no problem in assuming a synchronisation of events. Hence in this paper, we construct a probability distribution for the clock reading that are simultaneous with the realisation of a specific quantum event (usually a particle detection). It is important to emphasise that the determination of the time-of-arrival involves only *a single act of measurement*, which corresponds to the particle being registered by the detector.

In each individual run of the experiment we record the moment the particle entered the measuring device. The only way to do so is by checking at every single moment of time which of the two alternatives holds: the particle having been detected or not. We then identify the time-of-arrival as the moment of transition from the event of no-detection to the event of detection. For this purpose, we need to consider a (continuous) history of alternatives of detection. This suggests that a framework based on histories (path integrals or the consistent histories approach) is particularly suitable for that purpose. Indeed, an important feature of the histories formalism is that it distinguishes the role of the causal ordering of events in quantum theory from that of evolution (see [6] or the more general discussion in [7]). This allows us to set up the problem of the time-of-arrival probabilities at a kinematical level, i.e. without specific reference to the system's Hamiltonian.

A key feature of the constructions we present here is that they are not tied to a specific choice of the Hamiltonian that describes the quantum system's dynamics. From the experimentalist's point of view, this is perhaps the most natural procedure. One may measure the time-of-arrival for any system without any prior knowledge about the system's dynamics. All that is needed is a particle detector and an external clock. We would use the same detector for a free particle, for a particle moving in a potential, for a particle in presence of an environment, or even when we have complete ignorance about the particle's dynamical behaviour. The procedure one follows to measure the time-of-arrival should not depend in principle on the system's dynamics, only the results (namely the probabilities) should.

The histories description provides an important technical advantage. While the time-of-arrival does not define a function on the space of single-time alternatives of the system, it becomes one in the space of histories. In fact, the problem of defining a time-of-arrival probability is a special case of defining probabilities

for histories. We shall exploit this fact in section 5, in order to construct a Positive-Operator-Valued-Measure for histories, by mirroring the corresponding construction of probabilities in sequential measurements.

To demonstrate unambiguously that time-of-arrival is naturally defined in a histories framework, we consider an analogous construction, namely the time-of-arrival in stochastic processes (section 2). At this level a probability density for the time-of-arrival can be simply constructed. However, when we pass to quantum theory a problem appears. The probabilities obtained from Born's rule depend on the time variable in a way that does not allow the straightforward definition of a probability density. As a result, there is a strong ambiguity in the implementation of the continuous-time limit.

The second complication involved in quantum theory is the fact that the construction depend strongly on the interpretation of the "wave-packet reduction" rule. One possible interpretation is that the wave-packet reduction is a physical process that takes place only after the measured system has interacted with a measuring device. Another interpretation is that the reduction rule can be applied to any circumstance, in which we have obtained information about the quantum system. For example, if a particle detector did not click at a specific moment of time, then we can infer that the particle has been outside the region of detection at that time.

The first interpretation allows us to construct time-of-arrival probabilities by applying *classical reasoning* to the quantum mechanical probabilities. The corresponding probability density is not a linear functional of the initial density matrix, because the assumption we employed violates the "logic" of quantum mechanical propositions. Nonetheless, it has the correct classical limit and physically reasonable properties. Its main drawback is that it is ambiguous with respect to the continuous-time limit, and depends on the procedure one employs for its implementation.

The second interpretation of the reduction procedure lends itself to constructions that emphasize the 'logic' of quantum events. All information we obtain for a quantum system, whether this arises from a concrete experimental datum or from inference from the lack of such a datum (i.e. the detector not having clicked) are treated in the same footing. An important example is the consistent histories approach, which we examine in section 4. Time-of-arrival histories are a special case of the so-called "spacetime coarse-grainings" that have been studied before by Hartle [8] and others [9]. The advantage conferred to us by this approach is that the continuous-time limit *is naturally obtained at the level of amplitudes* (which essentially correspond to restricted path integrals). The problem arises at the level of combining the amplitudes in order to obtain a probability density. In effect, there is strong interference between different values of arrival times that are mutually exclusive in the classical context.

The quantum Zeno effect [10] poses another problem for the determination of probabilities; it can be partially evaded but it remains troublesome at the fundamental level. It seems that the only information we can obtain unambiguously

in this framework is the classical deterministic limit for the time-of-arrival.

In section 5, we apply the results obtained from the analysis of histories in a different context, namely the construction of a Positive-Operator-Valued-Measure that provides probabilities for the time-of-arrival. The key idea is that the construction of a probability density for the time of arrival is not fundamentally different from that of probabilities for sequential measurements. Hence, we follow a procedure developed for the study sequential measurements [11, 12]. The resulting time-of-arrival probabilities, like those of sequential measurements, are contextual; they depend strongly on the resolution of the measurement device. However, in the physically relevant regime for the free particle the dependence on τ drops out, and the constructed POVM coincides with that of Kijowski [4].

The approach to the time-of-arrival developed in Section 5 involves acting with the projection operator corresponding to no detection on the system's wavefunction at every moment of time. This action loosely corresponds to the fact that we have obtained information from the quantum system. It is important to emphasise that it does not refer to a physical act of measurement. It cannot be described in the language of standard quantum measurement theory: a von Neumann measurement, for example, involves a finite time interval during which the interaction Hamiltonian between system and measuring device dominates over the system's self-Hamiltonian. This is clearly not happening in the time-of-arrival set-up, at any moment prior to the system entering the measuring device. In quantum measurement theory, a physical measurement involves pre-measurement and reduction, and here the former part is missing. In physical terms, the time-of-arrival measurement involves a single measuring device, a single act of detection, a single irreversible change in the device and a single moment of time at which the interaction Hamiltonian becomes dominant. The only difference from the case of standard measurements is that the time of detection is unknown.

2 Time-of-arrival in stochastic processes

We first study the time-of-arrival probability in the theory of stochastic processes. This allows us to demonstrate the procedure we will follow in quantum theory, without the complications arising from the interpretations of the quantum measurement process.

We consider for simplicity a one-dimensional system, the state of which is fully specified at a moment of time by the position variable x . The sample space Ω is then identified with \mathbf{R} . An ensemble of such systems is described at $t = 0$ by the probability density $\rho_0(x)$. This probability density evolves according to the law

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho, \quad (2.1)$$

where \mathcal{L} is a linear, positive, trace-preserving operator on the space of probability densities.

We next construct the stochastic process corresponding to the system described by Eq. (2. 1). We assume that the measurements take place in the time interval $I = [0, T]$, where T may be eventually taken to infinity. The sample space Ω^I for the stochastic process is the space of all continuous paths $x(\cdot)$ from I to \mathbf{R} . The relevant random variables are the function X_t on Ω defined as $X_t(x(\cdot)) = x(t)$. There exists a probability measure on Ω^I given by the "continuum limit" of discrete-time paths $(t_1, x_1; t_2, x_2; \dots; t_n, x_n)$

$$d\mu_{\{t_1, t_2, \dots, t_n\}}(x_1, x_2, \dots, x_n) = \rho_0(x_0)g(x_0, 0; x_1, t_1)g(x_1, t_1; x_2, t_2) \dots g(x_{n-1}, t_{n-1}; x_n, t_n), dx_0 dx_1 dx_2 \dots dx_n, \quad (2. 2)$$

where $g(x, t; x', t')$ represents the propagator associated to equation (2. 1). We assume that ρ_0 has support only for values of $x < 0$.

From the probability measure (2. 2) we construct the probability for the proposition that the particle is detected at $x = 0$ at a specific time t , $0 < t < T$. For this purpose, we split the interval $[0, T]$ into N time steps of width $\delta t = T/N$. We assume also that $t = n\delta t$ for an integer $n < N$ and write $m = N - n$. We denote by χ_{\pm} the characteristic functions of the intervals $-\infty < x < 0$ and $0 < x < \infty$ respectively.

If the particle crosses the surface $x = 0$ for the first time within the time-interval $[t, t + \delta t]$, then it must have been in the region $(-\infty, 0)$ for all times less or equal to t and in the region $(0, \infty)$ at time $t + \delta t$. There is no reason to make any assumption about where it will be at times larger than $t + \delta t$, because in time-of-arrival measurements we are interested only in the time of the first detection. The particle may be, for example absorbed by the detector at time t .

With the above considerations in mind, we see that the probability that the particle is measured during the interval $[t, t + \delta t]$ equals

$$p(x = 0|[t, t + \delta t]) = \mu(D_{[t, t + \delta t]}), \quad (2. 3)$$

where μ is the stochastic probability measure and

$$D_{[t, t + \delta t]} = \chi_- \otimes \chi_- \otimes \dots \otimes \chi_- \otimes \chi_+ \otimes 1 \otimes \dots \otimes 1. \quad (2. 4)$$

The function $D_{[t, t + \delta t]}$ is a characteristic function on $\times_{t_i} \Omega_{t_i}$, and depends only on the value of n , namely the time-step that corresponds to detection. We may then also write $D_{[t, t + \delta t]}$ as D_n . If we also define by \bar{D} the characteristic function

$$\bar{D} = \chi_- \otimes \chi_- \otimes \dots \otimes \chi_- \quad (2. 5)$$

that corresponds to the particle never crossing $x = 0$ within the time interval $[0, T]$, the following relations hold

$$D_n D_m = D_n \delta_{nm} \quad (2. 6)$$

$$D_n \bar{D} = 0 \quad (2.7)$$

$$\sum_{n=0}^N D_n + \bar{D} = 1. \quad (2.8)$$

The variables D_n, \bar{D} then define an exclusive and exhaustive set of alternatives¹, hence the restriction of the probability measure to the algebra they generate defines a proper normalised probability measure for the time-of-arrival in discrete time.

We next construct the continuous limit of this probability as $N \rightarrow \infty$. We use an operator notation, representing the action of the integral kernel as $e^{\mathcal{L}t}$. Using Eq. (2.2) we write

$$p(x=0|[t, t+\delta t]) = \int dx [\chi_+ e^{\mathcal{L}\delta t} [\chi_- e^{\mathcal{L}\delta t}]^n \rho_0](x). \quad (2.9)$$

If we denote by K_t the limit

$$K_t = \lim_{n \rightarrow \infty} [\chi_- e^{\mathcal{L}t/n} \chi_-]^n, \quad (2.10)$$

we obtain

$$p(x=0|[t, t+\delta t]) = \delta t \int dx [\chi_+ \mathcal{L} \chi_-(x) K_t \rho_0](x) \quad (2.11)$$

The fact that the probability of the first crossing is proportional to δt implies that we can pass to the continuum limit defining a probability density on $[0, T]$

$$p(t|x=0) = \int dx [\chi_+ \mathcal{L} K_t \rho_0](x). \quad (2.12)$$

This probability density is not normalised to one as there is a no-zero residual probability $p(N)$ that the particle is not detected at all within $[0, T]$

$$\int_0^T dt p(t|x=0) = 1 - \int dx [K_T \rho_0](x) := 1 - p(N) \quad (2.13)$$

For generic initial states and dynamics the residual probability does not vanish as $t \rightarrow \infty$.

Using the probability density $p(t|x=0)$, we define the probability of detection within any interval $[t_1, t_2]$ by integrating $p(t|x=0)$ in this interval. The definition of average values of quantities is slightly more intricate. The sample space for the time of arrival (at the continuum limit) is not the interval $[0, T]$, but the set $[0, T] \cup \{N\}$, where N refers to the event of no detection. Strictly

¹This decomposition is a special case of the Path-Decomposition-Expansion and has been used in the context of restricted path-integrals in reference [19].

speaking, physical observables are functions on $[0, T] \cup \{N\}$. Hence there is an ambiguity in the definition of a function representing time t , because there is no natural numerical value it can take when evaluated on N .

In effect, a time function is defined unambiguously as a *conditional expectation*, namely after the assumption that the particle has actually been detected. This implies that we restrict (condition) the sample space to $[0, T]$. The conditional probability p_c density is then

$$p_c(t|x=0) = \frac{p(t|x=0)}{1 - p(N)}. \quad (2.14)$$

2.1 Examples

2.1.1 Two level system.

In the derivation of the probability of time-of-arrival, we referred to the variable x as position. However, the derivation is completely general and Eq. (3.15) may be applied to any sample space. If the latter is discrete, we have to exchange the integral with a summation. We may consider for example a stochastic two-level system (a bit) and determine the probability for the time-of-transition. In this case the sample space then consists of two alternatives: 0 and 1. We assume that initially the system is found at state 0. The most general operator \mathcal{L} consistent with positivity and normalisation of probabilities is

$$\mathcal{L} = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix} \quad (2.15)$$

The corresponding transition matrix for a small time interval δt is

$$\begin{pmatrix} 1 - a\delta t & b\delta t \\ a\delta t & 1 - b\delta t \end{pmatrix}, \quad (2.16)$$

which is the most general stochastic map for a two-level system.

It is then easy to compute the probability density for the time of the transition $0 \rightarrow 1$

$$p(0 \rightarrow 1; t) = be^{-bt}. \quad (2.17)$$

The probability $p(N)$ that no transition took place within the time interval $[0, T]$ equals e^{-bT} . As $T \rightarrow \infty$, $p(\infty) = 0$ and we may compute the mean time of transition

$$\langle t \rangle = b^{-1}, \quad (2.18)$$

which are the standard results for decay processes.

2.1.2 Wiener process

We next consider the case of the Wiener process, defined by the evolution operator

$$\mathcal{L}\rho = \frac{D}{2}\partial^2\rho, \quad (2.19)$$

where D is a diffusion constant. We assume that the particle is initially localised at $x = -L$, namely $\rho_0(x) = \delta(x + L)$.

The operator K_t is the propagator corresponding to \mathcal{L} with the Dirichlet boundary conditions at $x = 0$ ².

The integral kernel $K(x, x'; t)$ corresponding to K_t is

$$K(x, x'; t) = \chi_-(x)\chi_-(x')\sqrt{\frac{1}{2\pi Dt}}\left(e^{-(x-x')^2/2Dt} - e^{-(x+x')^2/2Dt}\right), \quad (2.20)$$

yielding

$$p(t|x=0) = \sqrt{\frac{1}{2\pi Dt}}\frac{L}{2t}e^{-L^2/2Dt}, \quad (2.21)$$

while

$$p(N) = \text{erf}(L/\sqrt{2DT}) - \text{erf}(-L/\sqrt{2DT}), \quad (2.22)$$

where erf is the error function.

3 The Copenhagen description

3.1 The standard construction and its problems

We next attempt to construct the time-of-arrival probability for quantum theory within the Copenhagen interpretation.

We split the time-interval $[0, T]$ into n time-steps of width $\delta t = T/n$. We represent the projection operators that correspond to the particle lying within $(-\infty, 0]$ and in $[0, \infty)$ as \hat{P}_- and \hat{P}_+ respectively. The time-of-arrival is defined as the moment the particle crosses from $(-\infty, 0)$ to $[0, \infty)$. We assume that at $t = 0$ the particle is described by a density matrix $\hat{\rho}_0$.

The probability that the particle crossed $x = 0$ at time t_1 equals $p_1 = \text{Tr}(\hat{\rho}e^{i\hat{H}t_1}\hat{P}_+e^{-i\hat{H}t_1})$. The probability that the particle crossed $x = 0$ at the next moment t_2 is then equal to

$$p_1 p(-, t_1; +, t_2), \quad (3.1)$$

²This is in fact a more general result. A quick but not fully rigorous way to see this is by writing the characteristic function of a set C as $\chi_C(x) = e^{-V_C(x)\delta t}$, where $V_C(x)$ is a "confining potential" that takes value 0 within C and ∞ outside C . We may then use the Trotter product formula $\lim_{n \rightarrow \infty} (e^{\mathcal{L}t/n}e^{-V_-t/n})^n = e^{(\mathcal{L}-V_-)t}$, which is exactly the propagator for \mathcal{L} with Dirichlet conditions at the boundaries of C .

where $p(-, t_1; +t_2)$ is the conditional probability that the particle was found within $(-\infty, 0]$ at t_1 and within $(0, \infty)$ at t_2 . According to the standard "reduction" rule this equals

$$p(-, t_1; +t_2) = \frac{\text{Tr}(\hat{P}_+ e^{-i\hat{H}(t_2-t_1)} \hat{P}_- e^{-i\hat{H}t_1} \hat{\rho}_0 e^{i\hat{H}t_1} \hat{P}_- e^{i\hat{H}(t_2-t_1)})}{\text{Tr}(\hat{\rho}_0 e^{i\hat{H}t_1} \hat{P}_- e^{-i\hat{H}t_1})}, \quad (3. 2)$$

yielding

$$p_2 = \text{Tr}(\hat{P}_+ e^{-i\hat{H}(t_2-t_1)} \hat{P}_- e^{-i\hat{H}t_1} \hat{\rho}_0 e^{i\hat{H}t_1} \hat{P}_- e^{i\hat{H}(t_2-t_1)}). \quad (3. 3)$$

Following the same procedure we obtain the probability that the particle crossed $x = 0$ at the k -th time step

$$p_k = \text{Tr}(\hat{P}_+ e^{-i\hat{H}(t_k-t_{k-1})} \dots \hat{P}_+ e^{-i\hat{H}(t_2-t_1)} \hat{P}_- e^{-i\hat{H}t_1} \hat{\rho}_0 \times e^{i\hat{H}t_1} \hat{P}_- e^{i\hat{H}(t_2-t_1)} \hat{P}_- \dots e^{i\hat{H}(t_k-t_{k-1})}). \quad (3. 4)$$

As we take the continuum limit $n \rightarrow \infty$, assuming that the initial state has support only in $(-\infty, 0)$, we obtain the probability that the particle was found between t and $t + \delta t$

$$p([t, t + \delta t]) = \delta t^2 \text{Tr}((\hat{C}_t \hat{H} \hat{\rho}_0 \hat{C}_t^\dagger \hat{H} \hat{P}_+), \quad (3. 5)$$

where

$$\hat{C}_t = (\hat{P}_- e^{-i\hat{H}t/n} \hat{P}_-)^n. \quad (3. 6)$$

There are two severe problems in this result. First, the probability $p([t, t + \delta t])$ is proportional to δt^2 , and hence does not define a probability density. If we tried to construct the probability for the detection within a finite time interval $[t_1, t_2]$ by integration, we would, strictly speaking, obtain zero, i.e. we would find that the particle is never detected.

Second, the probability that the particle never crosses $x = 0$ within the time interval $[0, T]$ equals

$$p(N) = \text{Tr}(\hat{C}_T \hat{\rho}_0 \hat{C}_T^\dagger). \quad (3. 7)$$

The operator \hat{C}_t is a degenerate unitary operator with a support in the range of \hat{P}_- [10]—this is the well-known quantum Zeno paradox. It follows that $\hat{C}_t \hat{C}_t^\dagger = \hat{P}_-$, hence

$$p(N) = \text{Tr}(\hat{\rho}_0 \hat{P}_-) = 1. \quad (3. 8)$$

It seems as though the particle can never cross $x = 0$, which is clearly a mistake. This problem can be addressed by noticing that the probabilities $p([t, t + \delta t])$ are by definition non-additive with respect to the projectors and

hence do not satisfy the Kolmogorov additivity condition, i.e. they are not probabilities at all. It is therefore no surprise that they do not define a probability density. One should employ for the continuum limit a probabilistic object that is additive with respect to the projectors. This is the decoherence functional of the consistent histories approach. The problem of the quantum Zeno effect is also partially alleviated in consistent histories, since the quantity $p(N)$ is not a genuine probability, unless a decoherence condition is satisfied. There are other problems, however, that appear in the implementation of the theory's classical limit. We will discuss this issue in more detail in the next section.

3.2 A different interpretation of the reduction procedure

An objection that can be raised to the derivation above has to do with the use of the conditional probability rule in Eq. (3. 2). One may argue that since the particle has not been detected at time t_2 , it has not interacted with the measuring device and for this reason, one should not employ the "reduction of the wave packet" rule, because no measurement has actually taken place³. Instead of the conditional probability, one should use the full probability of detection at t_2 , namely

$$Tr(\hat{P}_+ e^{-i\hat{H}t_2} \hat{\rho}_0 e^{i\hat{H}t_2}) := Tr(\hat{P}_+ \rho_{t_2}) \quad (3. 9)$$

hence the probability of detection at the k -th time step equals

$$p_k = (1 - \sum_{i=0}^{k-1} p_i) Tr(\hat{\rho}_{t_k} \hat{P}_+), \quad (3. 10)$$

This is a recursive equation with the following solution

$$\begin{aligned} p_k &= Tr(\hat{\rho}_{t_k} \hat{P}_+) \prod_{i=0}^{k-1} [1 - Tr(\hat{\rho}_{t_i} \hat{P}_+)] \\ &= Tr(\hat{\rho}_{t_k} \hat{P}_+) \exp \left[\sum_{i=0}^{k-1} \log(1 - Tr(\hat{\rho}_{t_i} \hat{P}_+)) \right]. \end{aligned} \quad (3. 11)$$

The expression above does not have a natural continuum limit. The sum in the exponential does not define an integral, because a δt term is missing. Again, we face the problem that the dependence of the quantum mechanical probabilities on time does not correspond to the existence of a genuine probability measure—hence a continuous limit does not exist naturally.

One way to obtain a probability measure is to introduce a time-step τ , which is a measure of the temporal resolution of the measuring device. For any time-scales much larger than τ one may substitute the sum in equation (3. 11) with

³See the discussion in the Introduction.

an integral, thus obtaining a probability measure on $[0, T]$

$$\begin{aligned} p(t) &= \frac{1}{\tau} \text{Tr}(\hat{\rho}_t \hat{P}_+) \exp\left[\frac{1}{\tau} \int_0^t ds \text{Tr}(\hat{\rho}_s \hat{P}_-)\right] = \\ &= -e^{t/\tau} \frac{d}{dt} e^{-F(t)}, \end{aligned} \quad (3. 12)$$

where $F(t) = \frac{1}{\tau} \int_0^t ds \hat{\rho}_s \hat{P}_-$.

The probability density above has a physically reasonable behaviour for a time-of-arrival probability. For a wave function, whose center follows approximately a classical path, the probability density (3. 12) is peaked around the classical time-of-arrival for this path. To see this one may consider Eq. (3. 12) for a free particle of mass m . Considering an initial Gaussian state

$$\psi_0(x) = \frac{1}{[\pi\sigma^2]^{1/4}} e^{-\frac{(x+L)^2}{2\sigma^2} + ipx} \quad (3. 13)$$

peaked at $t = 0$ around $x = -L$ with mean momentum p , we obtain for the function $F(t)$

$$F(t) = \frac{1}{2\tau} \int_0^t ds \left[1 + \text{Erf} \left(\frac{(L - \frac{p}{m}t)}{\sqrt{\sigma^2 + \frac{t^2}{m^2\sigma^2}}} \right) \right], \quad (3. 14)$$

which implies that (3. 12) has a strong peak around the classical time-of-arrival $t_{cl} = \frac{Lm}{p}$.

The problem lies in the strong dependence of these probabilities on the parameter τ . While it is reasonable to assume that the probabilities will be dependent on parameters that characterise the method of detection, we would intuitively expect that this dependence would be insignificant when we consider sufficiently large intervals of time. This is definitely not the case here as the probabilities are very sensitive on the value of τ .

Since quantum theory does not provide a natural way to pass into the continuum limit (at least in the scheme we consider in this section), it is natural to expect that different procedures will lead to different results. We may consider for example the following alternative.

In Eq. (3. 11) we may substitute in place of \hat{P}_+ a projector $\hat{P}_{\delta x}$ in position of width δx around $x = 0$, and the projector $\hat{1} - \hat{P}_{\delta x}$ in place of \hat{P}_- . This corresponds to a set-up by which the particle is detected only if it crosses the region $[-\delta x/2, \delta x/2]$. We next assume that the size δx decreases with δt , so that as $\delta t \rightarrow 0$, the area of detection also shrinks to zero. We therefore write $\delta x = v\delta t$, for some unspecified constant v with dimensions of velocity. This way of taking the limit essentially implies that actual detection of a particle needs a finite time-interval, since at the limit $\delta t \rightarrow 0$, $\hat{P}_{\delta x} = \hat{1}$.

Writing $\rho_t(0) = \langle x = 0 | \hat{\rho}_t | x = 0 \rangle$, we obtain at the limit $\delta t \rightarrow 0$ the probability that the particle is detected between time t and $t + \delta t$ as

$$p(t) \delta t = \delta t v \rho_t(0) e^{-v \int_0^t ds \rho_s(0)} = -\delta t \frac{d}{dt} e^{-v \int_0^t ds \rho_s(0)}. \quad (3. 15)$$

Hence, the probability that the particle is detected within the time interval $[t_1, t_2]$ equals

$$p([t_1, t_2]) = e^{-v \int_0^{t_1} ds \rho_s(0)} - e^{-v \int_0^{t_2} ds \rho_s(0)}, \quad (3. 16)$$

while the probability that the particle is not detected within the time-interval $[0, T]$ equals

$$p(N) = e^{-v \int_0^T ds \rho_s(0)}. \quad (3. 17)$$

The probability density (3. 15) has the correct behaviour at the classical limit, but again it depends on an unspecified parameter v , which this time has dimensions of velocity. One has to assume that v has to be identified with a characteristic property of the measuring device.

It follows that with the interpretation of the reduction rule we employed here, a probability distribution for the time-of-arrival cannot be constructed without making reference to the specific set-up through which it is determined. Whatever scheme one might employ, one has to introduce additional parameters in the description.

It would be mistaken, however, to consider the derivation leading to Eqs. (3. 15) or (3. 12) as inherently faulty. The only assumption we employed is that the reduction rule can only be applied, when an actual measurement has actually taken place, and not when we make an *inference* about the system by the fact that no detection has occurred. This implies, in particular, that the quantum Zeno effect is irrelevant for the time-of-arrival, because we do not have a continuous act of measurement (only a continuous inference). With this assumption the proof leading to (3. 11) only employs the classical rules of probability theory. In that sense, the key mathematical problem is that the dependence of the quantum mechanical probabilities on time does not allow the definition of a stochastic process—see the related discussion in [12]. Quantum probabilities *are not naturally densities with respect to time*; one can make them densities by introducing additional parameters.

It is important to note that this fundamental difficulty does not go away, when we enlarge the system by including degrees of freedom of the measurement device. The problem of finding a suitable continuous-expression for (3. 11) does not depend on specific features of the system's Hilbert space. The density matrix may include degrees of freedom of the measuring device or of an environment. The problem lies with the way time appears in the formalism of quantum theory.

In any case, equations (3. 15) and (3. 12) provide interesting candidates for a probability distribution for a time-of-arrival. They have a proper classical limit and they are mathematically unambiguous. In principle, they could be checked by any precision measurement of times-of-arrival.

4 The histories description

In this section, we follow a different approach from that of section 3.2. We assume that the reduction rule can be applied in any case we have obtained information about a quantum system. This allows us to preserve the ‘logical’ structure of quantum mechanical propositions. The natural scheme to explore the time-of-arrival problem is then the consistent histories approach. However, the results we obtain here are of a more general character. The mathematical objects employed in the consistent histories approach are essentially path-integrals and the amplitudes defined by these path integrals can be employed for the study of the time-of-arrival in different schemes. (We do that in section 5). The most important gain from this approach is that the continuous-time limit can be obtained unambiguously, because it is implemented at the level of amplitudes and not at that of probabilities.

4.1 Consistent histories

The consistent histories approach to quantum theory was developed by Griffiths [13], Omnés [14], Gell-Mann and Hartle [15, 16]. The basic object is a history, which corresponds to properties of the physical system at successive instants of time. A discrete-time history α will then correspond to a string $\hat{P}_{t_1}, \hat{P}_{t_2}, \dots, \hat{P}_{t_n}$ of projectors, each labelled by an instant of time. From them, one can construct the class operator

$$\hat{C}_\alpha = \hat{U}^\dagger(t_1)\hat{P}_{t_1}\hat{U}(t_1)\dots\hat{U}^\dagger(t_n)\hat{P}_{t_n}\hat{U}(t_n) \quad (4. 1)$$

where $\hat{U}(s) = e^{-i\hat{H}s}$ is the time-evolution operator. The probability for the realisation of this history is

$$p(\alpha) = Tr\left(\hat{C}_\alpha^\dagger\hat{\rho}_0\hat{C}_\alpha\right), \quad (4. 2)$$

where $\hat{\rho}_0$ is the density matrix describing the system at time $t = 0$.

However, the expression above does not define a probability measure in the space of all histories, because the Kolmogorov additivity condition cannot be satisfied: if α and β are exclusive histories, and $\alpha \vee \beta$ denotes their conjunction as propositions, then it is not true that

$$p(\alpha \vee \beta) = p(\alpha) + p(\beta). \quad (4. 3)$$

The histories formulation of quantum mechanics does not, therefore, enjoy the status of a genuine probability theory.

However, an additive probability measure *is* definable, when we restrict to particular sets of histories. These are called *consistent sets*. They are more conveniently defined through the introduction of a new object: the decoherence functional. This is a complex-valued function of a pair of histories given by

$$d(\alpha, \beta) = \text{Tr} \left(\hat{C}_\beta^\dagger \hat{\rho}_0 \hat{C}_\alpha \right). \quad (4.4)$$

A set of exclusive and exhaustive alternatives is called consistent, if for all pairs of different histories α and β , we have

$$\text{Re } d(\alpha, \beta) = 0. \quad (4.5)$$

In that case one can use equation (2.2) to assign a probability measure to this set.

4.2 Time-of-arrival histories

Histories and propositions about histories may be represented by projection operators on a Hilbert space $\mathcal{V} = \otimes_t H_t$, which the tensor product of the single time Hilbert spaces of standard theory—this is the History Projection Operator (HPO) formulation of the history theory [17]. The merit of this description is that the logical structure of history propositions is preserved (they form a lattice that corresponds with the lattice of subspaces of \mathcal{V}), and in the present context allows the arguments used for the time-of-arrival description of stochastic processes to be transferred into the quantum level. In particular, the continuum limit in time may be taken in an unambiguous manner. Note that in this scheme the decoherence functional is a Hermitian, bilinear functional on $\mathcal{V} \times \mathcal{V}$ that satisfies the following properties

$$d(1, 1) = 1 \quad (4.6)$$

$$d(0, \alpha) = 0 \quad (4.7)$$

$$d(\alpha, \alpha) \geq 0 \quad (4.8)$$

We next consider a description of time-of-arrival histories with n -time steps. One defines the projectors $\hat{\alpha}_m$ corresponding to the proposition that the particle crossed $x = 0$ for the first time between the m -th and the $m + 1$ -th time step

$$\hat{\alpha}_m = \hat{P}_- \otimes \hat{P}_- \otimes \dots \otimes \hat{P}_+ \otimes \hat{1} \otimes \dots \otimes \hat{1}, \quad (4.9)$$

as well as the projector $\hat{\alpha}$ corresponding to the proposition that the particle does not cross $x = 0$ within the n -time steps

$$\hat{\alpha} = \hat{P}_- \otimes \hat{P}_- \otimes \dots \otimes \hat{P}_- \quad (4.10)$$

Clearly these projectors satisfy

$$\hat{\alpha}_n \hat{\alpha}_m = \delta_{nm} \hat{\alpha}_n \quad (4.11)$$

$$\hat{\alpha}_n \hat{\hat{\alpha}} = 0 \quad (4.12)$$

$$\sum_m \hat{\alpha}_m + \hat{\hat{\alpha}} = 1. \quad (4.13)$$

Thus they form an exhaustive and exclusive set of histories, hence a sublattice of the lattice of history propositions⁴. One can therefore pullback the decoherence functional to this lattice, thereby obtaining a decoherence functional on a sample space consisting of the points (t_1, \dots, t_n) together with the point N corresponding to no crossing

$$d(t_n, t_m) = d(\hat{\alpha}_n, \hat{\alpha}_m) \quad (4.14)$$

$$d(N, t_n) = d(\hat{\hat{\alpha}}, \hat{\alpha}_n) \quad (4.15)$$

$$d(N, N) = d(\hat{\hat{\alpha}}, \hat{\hat{\alpha}}) \quad (4.16)$$

In analogy to the stochastic case, one may define a self-adjoint time-of-arrival operator \hat{T} on \mathcal{V} modulo its value on the subspace corresponding to $\hat{\hat{\alpha}}$, namely one may define

$$\hat{T}_{x=0} = \sum_i t_i \hat{\alpha}_i, \quad (4.17)$$

which is unambiguously defined on $\mathcal{V} - \text{Ran}(\hat{\hat{\alpha}}) - \text{Ran}(\hat{\hat{\alpha}})$ is the closed linear subspace corresponding to $\hat{\hat{\alpha}}$.

We next consider two discretisations $\{t_0 = 0, t_1, t_2, \dots, t_N = T\}$ and $\{t'_0 = 0, t'_1, t'_2, \dots, t'_{N'} = T\}$ of the time interval $[0, T]$ with time-step $\delta t = T/N$, and $\delta t' = T/N'$. We construct the decoherence functional $d([t, t + \delta t], [t', t' + \delta t'])$, where $n = tN/T$ and $m = t'N'/T$. This reads

$$d([t, t + \delta t], [t', t' + \delta t']) = \text{Tr} \left(\hat{\rho}_0 [e^{i\hat{H}\delta t'} \hat{P}_-]^n e^{i\hat{H}\delta t'} \hat{P}_+ \right. \\ \left. \times e^{i\hat{H}(t'-t)} \hat{P}_+ e^{-i\hat{H}\delta t} [\hat{P}_- e^{-i\hat{H}\delta t}]^m \right). \quad (4.18)$$

⁴One should note that the n -time histories we study here should not be viewed as discretisations of continuous time paths, but as histories corresponding to genuinely discrete time. The consideration of discretized alternatives in a continuous-time theory is conceptually problematic because at any time between t_i and t_{i+1} the particle may have crossed $x = 0$, and this fact will not be captured in the resulting propositions. Our approach is that we first consider alternatives of detection in a discrete-time theory, and we then identify a suitable continuous limit for the decoherence functional. This involves a choice on the way we *define* the continuous histories. This choice allows us to recover known results [8, 22]. However, this procedure may not be unique—see the discussion on alternative treatments of the quantum Zeno effect.

We take then the limit $N, N' \rightarrow \infty$, while keeping t and t' fixed. Assuming that ρ_0 lies within the range of \hat{P}_- , i.e. $\hat{P}_- \hat{\rho}_0 \hat{P}_- = \hat{\rho}_0$ we obtain

$$d([t, t + \delta t], [t', t' + \delta t']) = \delta t \delta t' \text{Tr} \left(e^{-i\hat{H}(t'-t)} \hat{P}_+ \hat{H} \hat{P}_- \hat{C}_t^\dagger \hat{\rho}_0 \hat{C}_{t'} \hat{P}_- \hat{H} \hat{P}_+ \right) \quad (4. 19)$$

where $\hat{C}_t = \lim_{n \rightarrow \infty} (\hat{P}_- e^{-i\hat{H}t/n} \hat{P}_-)^n$. Writing

$$\rho(t, t') = \text{Tr} \left(e^{-i\hat{H}(t'-t)} \hat{P}_+ \hat{H} \hat{P}_- \hat{C}_{t'}^\dagger \hat{\rho}_0 \hat{C}_t \hat{P}_- \hat{H} \hat{P}_+ \right) \quad (4. 20)$$

we see that the decoherence functional corresponds to a complex-valued density on $[0, T] \times [0, T]$. The additivity of the decoherence functional (which reflects the additivity of quantum mechanical amplitudes) allows us to obtain a continuum limit, something that could not be done if we worked at the level of probabilities. Consequently, one may obtain its values on coarse-grained histories corresponding to time-of-arrival lying within the subsets $[t_1, t_2]$ and $[t'_1, t'_2]$ of $[0, T]$ by integrating over $\rho(t, t')$

$$d([t'_1, t'_2], [t_1, t_2]) = \int_{t_1}^{t_2} dt \int_{t'_1}^{t'_2} dt' \rho(t, t') \quad (4. 21)$$

We then obtain the values of the decoherence functional for any pair of measurable subsets of $[0, T]$. However, the decoherence functional on $[0, T]$ is not properly normalised, because the actual space of time-of-arrival propositions is not the space of subsets of $[0, T]$, but rather the space of subsets of $[0, T] \cup \{N\}$, where N corresponds to the event of no detection. The values of the decoherence functional, when at least one of its entries is N are easily computed

$$d([t_1, t_2], N) = \int_{t_1}^{t_2} dt \text{Tr} \left(e^{-i\hat{H}(T-t)} \hat{C}_T^\dagger \hat{\rho}_0 \hat{C}_t \hat{P}_- \hat{H} \hat{P}_+ \right) \quad (4. 22)$$

$$d(N, N) = \text{Tr}(\hat{C}_T^\dagger \hat{\rho}_0 \hat{C}_T) = 1 \quad (4. 23)$$

The last equation is due to the fact that the operator \hat{C}_t is a degenerate unitary operator with support on the range of \hat{P}_- (the quantum Zeno effect).

The normalisation condition for the decoherence functional implies that

$$d([0, T], [0, T]) + d([0, T], N) + d(N, [0, T]) + d(N, N) = 1, \quad (4. 24)$$

which leads to

$$d([0, T], [0, T]) = -2\text{Re} d([0, T], N) \quad (4. 25)$$

It is important to note that in the context of consistent histories the fact that $d(N, N) = 1$ does not imply that the event N (never crossing $x = 0$) will be realised, because $d(N, N)$ does not correspond to a probability, unless

the consistency condition $Re\,d[0, T], N) = 0$ is satisfied. In this case, however, $d([0, T], [0, T]) = 0$ and hence that crossing $x = 0$ never takes place. This implies that the event of the particle not crossing the surface $x = 0$ can only be a member of a consistent set, in which the probability for crossing $x = 0$ is zero. This is rather counterintuitive, because it fails to give a correct classical limit—see related discussion in [21]. One would expect that at some level of coarse-graining one would obtain the classical result, namely a probability distribution sharply peaked around the classical time of arrival t_{cl} , similar to the one we derived in the last section.

Possible treatments of the quantum Zeno effect. A key feature of the quantum Zeno effect is that it is not robust. When one employs a positive operator \hat{E} in the definition of the operator \hat{C}_t (instead of a projector), the result is no more a degenerate unitary operator. This is true even if the operator \hat{E} is very close to a true projector $Tr|\hat{E}^2 - \hat{E}| = \epsilon/Tr\hat{E}$, for a number $\epsilon \ll 1$. In that case the matrix elements of \hat{C}_t fall with $e^{-\epsilon t}$, as we demonstrate in a simple example in the Appendix. This implies that even a very small deviation from a true projector leads to a qualitatively different result.

If our calculation of probabilities refers to actual measurements then the quantum Zeno effect should not pose a problem. Realistic measurements should be represented by POVMs rather than projection-valued-measures, in which case the quantum Zeno effect does not arise. However, this would spoil the continuous-time limit, which depends crucially on the fact that the operators \hat{P}_+ and \hat{P}_- correspond to exclusive alternatives. A naive substitution of the operators \hat{P}_+ and \hat{P}_- , by partially overlapping approximate projectors would introduce extra terms in the expression of the decoherence functional. It is easy to verify that these terms would be of the order of $||\hat{P}_+\hat{P}_-||$ and not dependent on δt , hence they would remain non-zero even at the limit $\delta t \rightarrow 0$. One could entertain the possibility that the approximate projectors could be dependent on δt , and that at the limit of δt their overlap becomes zero, i.e. they become true projectors. We have explored this possibility, but it does not seem to work. The Zeno effect still persists at the continuum limit. (This can easily be seen in the example we provide in the Appendix.)

While the continuous-time limit we constructed here leads invariably to a quantum Zeno effect, this is not the only way that this limit can be taken in the histories formalism. The construction of the decoherence functional we presented here is obtained from a limit of discrete time expression. We have not constructed actual continuous time histories and defined the decoherence functional straightforwardly on them. To do that one should proceed in the logic of the HPO approach and construct a history Hilbert space that would correspond in some sense to a "continuous-tensor product" of single-time Hilbert spaces. Such Hilbert spaces have been constructed before [18, 6]; they are not genuine continuous tensor products, but they share many of their features, and

they are obtained from group-theoretical arguments. A key property of this construction is that propositions have support on finite time-intervals $[t_1, t_2]$ and not on sharp points of the real line. The operator structure is then quite different at the kinematical level, and it raises the possibility that one could define a decoherence functional as a genuinely continuum object, in a way that does not suffer from the Zeno effect. For example, it is plausible that the operators entering the definition of the operator \hat{C}_t as the limit of $\delta t \rightarrow 0$, should also be dependent on δt , as they should refer to finite intervals of the real line rather than sharp points. As we argued earlier even a small change might be sufficient to remove the undesirable properties of \hat{C}_t ; the real issue is to justify such changes from first principles within the continuous-time-histories formalism. We shall elaborate on this construction in a follow-up paper.

Conditioning. As we showed in section 2, it is possible in classical probability to reduce the probability measure from the full algebra of subsets in $[0, T] \times \{N\}$ to the algebra of events on $[0, T]$. This reduction results from the use of conditional probability. We defined a probability for the time-of-arrival conditioned upon the premise that the particle did cross $x = 0$ at some time within the interval $[0, T]$.

This reasoning may be partially transferred to the quantum case. We cannot speak, however, for a conditional probability because this involves the consideration of consistent sets. Classical conditional probability is defined through a natural mathematical procedure, which employs the additivity of the probability measure over the space of functions on the sample space, to reduce the level of description into a subalgebra of events. Quantum probabilities are not additive over the lattice of events, but the decoherence functional is. In reference [20] the procedure of conditioning at the level of decoherence functional has been developed in detail, by generalising the classical notion of conditional expectation to the quantum level. One may define a decoherence functional over a subalgebra of events (namely propositions about histories), thus incorporating any information we may have obtained for the system. For the details of the procedure we refer the reader to [20], but for the simple case that the subalgebra with respect to which we implement the conditioning is generated by a single history proposition β , such that $d(\beta, \beta) \neq 0$, the conditioned decoherence functional is given by the intuitively simple expression

$$d_c(\alpha, \alpha') = \frac{d(\alpha \wedge \beta, \alpha' \wedge \beta)}{d(\beta, \beta)} \quad (4.26)$$

The resulting decoherence functional is the mathematical object that should be used in the derivation of probabilities, provided we know that events corresponding to the subalgebra have been realised.

In the present case, we need to condition the decoherence functional from the algebra of events corresponding to the sample space $[0, T] \times \{N\}$ to the

subalgebra of events corresponding to a sample space $[0, T]$, namely assuming that the particle has actually been detected. Since this operation involves "discarding" only a simple point of the sample space, the result is very simple. The conditioned decoherence functional d_c is obtained by a conditioned density

$$\rho_c(t, t') = \frac{\rho(t, t')}{d([0, T], [0, T])} = \frac{\rho(t, t')}{-2Re d([0, T], N)}. \quad (4. 27)$$

Since the event of no detection is removed from the resulting subalgebra, we may pretend that we have avoided the quantum Zeno effect. This is, however, an evasion and not a solution to the problem. It only allows one to differentiate the problem of defining a probability for the time-of-arrival, from the more general issue of properly defining a continuum limit, that avoids the quantum Zeno effect.

4.3 The free particle

For the simple case of a particle at a line with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x})$, where the potential is bounded from below, we may employ a result in [8, 22] that the restricted propagator \hat{C}_t is obtained from the Hamiltonian \hat{H} by Dirichlet boundary conditions⁵. If we also denote by $G_0(x, x'|t)$ the full propagator in the position basis (corresponding to $e^{-i\hat{H}t}$), we obtain

$$\rho(t, t') = \frac{1}{4M^2} \partial_x (\hat{C}_{t'} \psi_0)^*(0) \partial_x (\hat{C}_t \psi_0)(0) G_0(0, 0|t' - t) \quad (4. 28)$$

$$d(t, N) = -\frac{1}{2M} \int_{-\infty}^0 dx (\hat{C}_T \psi_0)^*(x) \partial_x (\hat{C}_t \psi_0)(0) G_0(x, 0|T - t), \quad (4. 29)$$

where $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$, with ψ_0 having support in $(-\infty, 0]$.

Note that in the derivation of the equations above, the derivative ∂_x arises from the presence of a term $\hat{P}_+ \hat{H} \hat{P}_-$ in the operator product in Eq. (4. 20). The contribution of the potential $V(x)$ vanishes, and the only contribution comes from the \hat{p}^2 of the kinetic energy.

For a free particle of mass M

$$G_0(x, x', t) = \sqrt{\frac{M}{2\pi i t}} e^{iM(x-x')^2/2t} \quad (4. 30)$$

$$C_t(x, x') = \chi_-(x) \chi_-(x') \sqrt{\frac{M}{2\pi i t}} \left[e^{iM(x-x')^2/2t} - e^{iM(x+x')^2/2t} \right] \quad (4. 31)$$

leading to

$$\partial_x (\hat{C}_t \psi_0)(0) = \sqrt{\frac{M}{2\pi i t}} \int_{-\infty}^0 dx \frac{-2iMx}{t} e^{i\frac{Mx^2}{2t}} \psi_0(x) := -2\partial_x (\hat{U}_t \psi_0)(0), \quad (4. 32)$$

⁵The result cited is valid for bounded intervals of the real line; the generalisation to semi-bounded intervals however is straightforwardly obtained using their method.

where we assumed that $\hat{P}_- \psi_0 = \psi_0$.

It follows that

$$\rho(t, t') = (\frac{\hat{p}}{M} \hat{U}_t \psi_0)(0) (\frac{\hat{p}}{M} \hat{U}_{t'} \psi_0)^*(0) \sqrt{\frac{M}{2\pi i(t' - t)}} \quad (4.33)$$

4.4 The classical limit

To verify that the decoherence functional for the free particle has the correct classical limit we consider a Gaussian initial state centered around $x = -L$ and with mean momentum equal to p

$$\psi_0(x) = (\pi\sigma_0^2)^{-1/4} e^{-(x+L)^2/2\sigma_0^2 + ipx}. \quad (4.34)$$

This state is localised within $[-\infty, 0]$ within an error of order e^{-L^2/σ^2} . We then obtain

$$\partial_x(\hat{C}_t \psi_0)(0) = \frac{2p}{\pi^{1/4}} \sqrt{\frac{\sigma_0}{\sigma^2(t)}} \left(\frac{t - t_{cl}}{M\sigma^2(t)} + i \right) e^{-\left(\frac{p^2}{2m^2\sigma^2(t)}\right)(t - t_{cl})^2 - i\frac{p^2}{M}(t - t_{cl})} \quad (4.35)$$

where

$$\sigma^2(t) = \sigma_0^2 \left(1 + i \frac{t}{M\sigma_0^2} \right) \quad (4.36)$$

and $t_{cl} = \frac{LM}{p}$ is the classical time-of-arrival.

Choosing T very large ($T \rightarrow \infty$), so that the classical time of arrival lies well within $[0, T]$ we see that the bi-density $\rho(t, t')$ has a singularity for $t = t'$ and that it is sharply peaked in each of its arguments around $t_{cl} = \frac{LM}{p}$, with a width δ of the order of

$$\delta = \frac{M|\sigma(t_{cl})|}{p} = \frac{M\sigma_0}{p} \sqrt{1 + \frac{L^2}{\sigma_0^4 p^2}}. \quad (4.37)$$

Note that for large values of L/p the width δ also becomes very large. This is due to the fact that the free time evolution causes the wave packet to spread in time. A very small value of σ_0 (hence a large initial momentum uncertainty) leads to large values of δ .

Hence if we consider a coarse-grained history α_{cl} for the time of crossing, which is centered around t_{cl} and has a width of $\Delta t \gg \delta$, and we denote as α'_{cl} the complement of α_{cl} , we obtain for the conditioned decoherence functional

$$|d_c(\alpha_{cl}, \alpha'_{cl})| = O(e^{-(\Delta t/\delta)^2}) \quad (4.38)$$

$$d_c(\alpha_{cl}, \alpha_{cl}) = 1 - O(e^{-(\Delta t/\delta)^2}), \quad (4.39)$$

hence we conclude that history α_{cl} will almost definitely be realised, provided the particle crosses $x = 0$ at some time within $[0, T]$.

Clearly there exists no classical limit, if either the initial state is not sufficiently localised in position, or if it is too localised so that the momentum uncertainty is very large, or if it is a superposition of states with distinct value of momentum.

4.5 Inclusion of measurement device

In the consistent histories interpretation probabilities are only defined, if the consistency condition is satisfied. For time-of-arrival propositions this happens for coarse-grained histories like α_{cl} of Sec. 4.4, which essentially correspond to the classical result. (Note that this result is obtained after conditioning the decoherence functional upon arrival). However, consistent histories refer to closed systems—hence, to obtain a prediction for the time-of-arrival probabilities in the general case we have to model the interaction of the particle with the measuring device.

There are various models for such interaction with various degrees of complexity [23] (or for a more general case see [24]). We consider here a very simple one, which will allow us to analyse some basic features of this procedure. We model the pointer of the measuring device with a two level system. The pointer is found in the state of lower energy, while the state of higher energy corresponds to the detector having clicked. The combined Hamiltonian of the system+apparatus is then

$$\hat{H} = \frac{\Omega}{2}(1 - \sigma_3) + \hat{H}_0 + \epsilon \hat{P}_+ \sigma_1, \quad (4.40)$$

where \hat{H}_0 is the particle's Hamiltonian, and the interaction characterised by a coupling constant ϵ is switched on, only when the particle enters the region $x > 0$. We next construct histories similar to the ones of section 4, only that they would refer to the properties of the pointer, i.e. they would be constructed from projectors of the form

$$\hat{E}_+ = \hat{1}_{particle} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \hat{E}_- = \hat{1}_{particle} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.41)$$

inserted into the definition of histories of the form (4.9-4.10). Hence we seek the moment of transition from the lowest energy state of the detector to the higher energy state.

We assume that the initial state is factorised $\hat{\rho} = \hat{\rho}_0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where $\hat{\rho}_0$ is the particle's density matrix. Using similar arguments we obtain the following expressions for the densities defining the decoherence functional

$$\rho(t, t') = \epsilon^2 Tr(e^{-i\hat{H}_0(t'-t)} \hat{P}_+ e^{-i\hat{H}_0 t} \hat{\rho}_0 e^{i\hat{H}_0 t'} \hat{P}_+) + O(\epsilon^4) \quad (4.42)$$

$$d(t, N) = \epsilon Tr(e^{-i\hat{H}_0(T-t)} \hat{P}_+ e^{-i\hat{H}_0 t} \hat{\rho}_0), \quad (4.43)$$

while $d(N, N) = 1$. Again we can define the conditioned decoherence functional on $[0, T]$ as $\rho_c(t, t') = \rho(t, t')/d([0, T], [0, T])$.

This model does not solve the fundamental problem of defining a probability density—one can easily check that the consistency condition is approximately satisfied only for highly coarse-grained histories centered around the classical time-of-arrival. On one hand, this is not a surprise. Realistic measurement devices are much more complex than the system described by the Hamiltonian (4.40). Moreover, they involve by necessity a degree of irreversibility, which is incompatible with a unitary evolution law (see for example a model in [25]): the states of detection and no-detection are asymmetric, because the latter involves an amplification procedure that leads to a macroscopic designation that the particle has been detected. Such an asymmetry is incompatible with unitary evolution.

On the other hand, the present model demonstrates a rather generic feature associated with the measurement problem, that makes its present felt in all interpretations of quantum theory that attempt to describe the measurement process unitarily. In the consistent histories approach measurement devices are thought of as quantum systems, which are characterised by a consistent set of histories for the pointer device, so that the values of the pointer can be ascertained individually (and assigned probabilities) for a large class of initial states of the measured system [14, 16]. This is equivalent with obtaining a density matrix diagonalised in a basis factorised with respect to the degrees of freedom of the system and the measuring device, which necessary, in order to attribute definite values to the macroscopic pointer.

In the model we presented here, this is clearly not the case. There exist also very general theorems that state that such a factorisation is in general not possible [27], see also the discussion in [28, 29, 30] and in a different but related context the theorems of [31]). The general argument is rather simple and in the present context takes the following form. The derivation of the decoherence functional for time-of-arrival histories remains the same, whether the Hilbert space is that of a single quantum system, or if it includes the degrees of freedom of a measurement device or of an environment. The non-zero value of the off-diagonal elements of the decoherence functional would then still persist, except possibly for the case of sufficient coarse-graining corresponding to the classical results. Hence, it seems that, unless we introduce additional assumptions, the consistent histories scheme cannot produce more detailed information about the time-of-arrival probability, beyond the determination of the classical limit.

The situation is different when the evolution of the particle (rather than that of the measurement device) involves a degree of irreversibility [26]. Indeed, in the presence of a decohering environment the evolution of the particle is closely approximated (after a typically short decoherence time) by a stochastic process, in which case one may employ the construction of time-of-arrival probabilities

sketched in section 2. However, the time-of-arrival seems to depend rather crucially on properties of the environment, which seems to destroy much "more" interference than what is necessary to define a consistent set of histories⁶

Another possibility would be to consider a different measuring apparatus interacting with the particle at each moment of time. This, however, would describe a different physical circumstance than the one we consider in this paper. Here we consider the case that the only information we can obtain from the system is a signal that the particle has been detected at a specific moment of time, i.e. there is a single event of measurement. Inserting multiple measuring devices would be equivalent to letting the particle move within a material that records its track (like a bubble chamber). In that case our datum is a continuous history of the particle, and can be treated within the theory of continuous time measurement. Such an interaction inevitably causes the particle to decohere, and as such its evolution can be well approximated by a stochastic process. The derivation of the time of arrival would then be much simpler in this case and would follow the general scheme of [26]. This is, however, different from the issue taken up in this paper.

5 An operational description for time-of-arrival measurements

The derivation of the decoherence functional for the time-of-arrival propositions does not answer the question we asked at the beginning of this paper: it is in principle possible to measure the time-of-arrival in individual runs of an experiment, therefore constructing the relative frequencies for a detection of the particle within any time interval $[t_i, t_f]$. From these frequencies of events we can construct in the limit of a large number of runs a probability for the detection of the particle within any time interval. One then is entitled to ask, how to obtain this operationally meaningful probability density from the rules of quantum theory.

The main contribution of the histories formalism to the final result is that it allows one to implement the continuous-time limit. However, this passage refers not to a probability density, but to the decoherence functional and one needs further assumptions in order to construct genuine probabilities. As the expression (4. 20) for the decoherence functional demonstrates, there is persistent interference between the different alternatives for the time-of-arrival.

⁶There is a rather paradoxical situation in the presence of environment. If the full quantum mechanical treatment of system+environment is taken into account, in which case the full evolution law is unitary, we are faced with the quantum Zeno effect. Any proposition about the system will be represented by projection operators, and the operator \hat{C}_t will still be a degenerate unitary operator. If, however, one describes the effect of the environment in terms of a non-unitary evolution (or a stochastic process), an excellent approximation in many cases, no such problem seems to arise. This suggests again that the quantum Zeno effect is not robust.

This problem is not specific to the time-of-arrival measurements. It is a special case of a more general problem, that of defining a probability density for the outcomes of measurement that take place at more than one moment of time. This issue has been analysed in [11, 12]. In sequential measurements it is possible to obtain the probabilities in terms of a Positive-Operator-Valued-Measure, whose mathematical form is markedly similar to the "probabilities" constructed by the consistent histories approach. We shall attempt to do the same here for the time-of-arrival probabilities, thus exploiting the convenient continuous-time limit incorporated in the decoherence functional.

Note that the considerations in this section are purely operational and employ the Copenhagen interpretation: we do not consider closed, individual systems, but are only interested about probabilities obtained in specific measurement situations, which we assume to refer to a statistical ensemble. Hence even though we shall use the mathematical apparatus of consistent histories, the context in which we work is markedly distinct.

5.1 Probabilities for sequential measurements

Discrete spectrum Let us consider the two-time measurement of an observable $\hat{x} = \sum_i \lambda_i \hat{P}_i$ with discrete spectrum. Writing $\hat{Q}_i = e^{i\hat{H}t} \hat{P}_i e^{-i\hat{H}t}$, we construct the probabilities for the most-fine grained two-time results

$$p(i, 0; j, t) = \text{Tr}(\hat{Q}_j \hat{P}_i \hat{\rho}_0 \hat{P}_i) = |\langle \hat{\rho}_0 | i \rangle|^2 |\langle i | e^{-i\hat{H}t} | j \rangle|^2 \quad (5. 1)$$

Irrespective of the interpretation of the measurement process, the probabilities (5. 1) refer to the most elementary alternatives that can be unambiguously determined in the experimental set-up corresponding to the sequential measurement of \hat{x} . Therefore, they can be employed to construct probabilities for general sample sets U_1, U_2 on the spectrum Ω of \hat{x} , namely

$$p(U_1, 0; U_2, t) = \sum_{i \in U_1} \sum_{j \in U_2} p(i, 0; j, t). \quad (5. 2)$$

The total probability is normalised

$$p(\Omega, 0; \Omega, t) = \sum_{ij} \text{Tr}(\hat{Q}_j \hat{P}_i \hat{\rho}_0 \hat{P}_i) = 1, \quad (5. 3)$$

a property that follows from the fact that $\sum_i \hat{P}_i = \sum_j \hat{Q}_j = \hat{1}$.

Eq. (5. 2) defines a POVM for two-time measurements. Note the difference from equation (4. 2). Equation (4. 2) is valid only for the most fine-grained alternatives. Any further coarse-graining is done by summing only the elementary probabilities that correspond to the value of the decoherence functional for the most fine-grained histories. This result implies that we can use the decoherence functional to construct POVMs for sequential measurements and may expect to repeat do the same for the case of time-of-arrival.

Continuous spectrum When we consider the case of an operator with discrete spectrum a problem appears. There are no fine-grained projectors and the choice of the elementary quantum probabilities, from which one may build the general probabilities for measurement outcomes cannot be made uniquely.

The immediate generalisation of Eq. (5. 1) for the measurement of an operator with a continuous spectrum is

$$p(x_1, 0; x_2, t) = |\langle x_1 | \hat{\rho}_0 | x_1 \rangle|^2 |\langle x_1 | e^{-i\hat{H}t} | x_2 \rangle|^2. \quad (5. 4)$$

However, this does not define a proper probability density, because it is not normalised to unity

$$\int dx_1 \int dx_2 p(x_1, 0; x_2, t) = \infty. \quad (5. 5)$$

This is due to the fact that there can be no measurements of infinite accuracy. One has, therefore, to take into account the finite width of any position measurement, say δ . This quantity depends on the properties of the measuring device—for example the type of the material that records the particle's position.

The simplest procedure is to consider the measurement of a self-adjoint operator $\hat{x}_\delta = \sum_i x_i \hat{P}_i^\delta$, where \hat{P}_i^δ is a projection operator corresponding to the interval $[x_i - \frac{\delta}{2}, x_i + \frac{\delta}{2}]$. In that case we may immediately construct the fine-grained probabilities

$$p_\delta(i, 0; j, t) = Tr(\hat{Q}_j^\delta \hat{P}_i^\delta \hat{\rho}_0 \hat{P}_i^\delta), \quad (5. 6)$$

from which we may construct probabilities for general sample sets U_1 and U_2 :

$$p_\delta(U_1, 0; U_2, t) = \sum_{i \in U_1} \sum_{j \in U_2} Tr(\hat{Q}_j^\delta \hat{P}_i^\delta \hat{\rho}_0 \hat{P}_i^\delta). \quad (5. 7)$$

Strictly speaking one may only consider sample sets that are unions of the elementary sets that define our lattice. If, however, the size of the sample sets L is much larger than δ , we may approximate the summation with an integral. This amounts to defining the continuous version of probabilities (5. 6)

$$p_\delta(x_1, t_1; x_2, t_2) = Tr \left(e^{i\hat{H}(t_2-t_1)} \hat{P}_{x_2}^\delta e^{-i\hat{H}(t_2-t_1)} \hat{P}_{x_1}^\delta \hat{\rho}(t_1) \hat{P}_{x_1}^\delta \right), \quad (5. 8)$$

where we denoted $\hat{P}_x^\delta = \int_{x-\delta/2}^{x+\delta/2} dy |y\rangle \langle y|$.

The important result of this analysis is that the probabilities for sequential measurements (5. 6) depend strongly on the resolution δ of the measuring device. This dependence is very strong: even probabilities of sample sets coarse-grained at a scale much larger than δ exhibit a very strong dependence on δ . From a mathematical point of view this dependence is a consequence of the

fact that the off-diagonal elements of the decoherence functional between fine-grained multi-time measurement outcomes do not vanish and are generically of the order of magnitude of the probabilities themselves. Hence, when we compare a probability corresponding to a value δ , with another one corresponding to 2δ , they differ by an amount proportional to the corresponding off-diagonal terms of the decoherence functional, which is in general substantially large. The reader is referred to [12] for extensive discussion and generalisations.

The construction probabilities for the outcomes of sequential measurements consists of two steps. First we identify the most fine-grained alternatives compatible with the measuring device at hand and we construct the corresponding elementary quantum probabilities by using the rule (4. 2). These fine-grained alternatives (referred to by the index a) correspond to specific functions $F_a[x(\cdot)]$ on the space of paths Ω^I . The elementary probabilities will be

$$p(a) = d(F_a, F_a). \quad (5. 9)$$

In the equation above the decoherence functional is viewed as a bi-linear functional on Ω^I . For example, in the two-time measurement of position the functions take the form $F_{ij}[x(\cdot)] = \chi_i^\delta(X_{t_1})\chi_j^\delta(X_{t_2})$, and

$$p_\delta(i, 0; j, t) = d(F_{ij}, F_{ij}), \quad (5. 10)$$

The next step involves the summation over those elementary probabilities to construct an additive measure that assigns probabilities to every sample set obtained by the coarse-graining of the elementary alternatives. We shall apply this procedure to the construction of time-of-arrival probabilities.

5.2 POVMs for time-of-arrival probabilities

Our contention is that the analysis of sequential measurements above may be transferred to the case of time-of-arrival measurements, because they share the crucial feature that they do not refer to the properties of a physical system at a single moment of time. This implies that the decoherence functional (4. 20) may be employed for the construction of a POVM on $[0, T]$ for the time of arrival probabilities, in analogy to that of (5. 6).

The diagonal elements $\rho(t, t)\delta t^2$ of the decoherence functional (4. 20) is essentially the modulus square of the amplitude that is obtained by the sum over all paths that cross the surface $x = 0$ within the interval $t + \delta t$. While the amplitude is obtained unambiguously through path integrals, its square cannot define a proper probability density, because of the presence of a term δt^2 rather than a δt one, but also because $\rho(t, t)$ diverges⁷. This divergence is analogous to that of (5. 4) for sequential measurements of position, and can be

⁷It is interesting to note that the δt^2 dependence disappears if the decoherence condition for histories holds—see [21].

removed in a similar manner by assuming a finite temporal resolution τ . Hence, we consider elementary intervals $[t_i, t_{i+1}]$ of width τ , τ corresponding to the temporal resolution of our measurement device. The elementary probabilities will be

$$\rho_i^\tau = \int_0^T dt \int_0^T dt' \rho(t, t') \chi_i^\tau(t) \chi_i^\tau(t'), \quad (5. 11)$$

where χ_i^τ is the characteristic function of the set $[t_i, t_{i+1}]$. One then may employ these probabilities to construct any probability corresponding to a set U constructed from the elementary cells $[t_i, t_{i+1}]$. By definition $\sum_i \chi_i^\tau = \chi_{[0, T]}$, hence the set of all i together with the event of no detection form a proper resolution of the unity.

The reader may object at this point that this leads us back to the discrete-time expression for the diagonal elements of the decoherence functional. This is not the case, because the probabilities (5. 11) involve the sum over *all* continuous paths that are detected in the time interval $[t_i, t_{i+1}]$ —hence it involves the contribution of any discretisation within $[t_i, t_{i+1}]$.

It is more convenient to avoid the discretisation procedure and construct a POVM on the continuous sample space $[0, T]$ —see ([12]) for the analogous procedure in sequential measurements. For this purpose we introduce a family of smeared delta functions $f_\tau(s, s')$ characterised by the parameter τ , which satisfy the following properties

$$\int_0^T ds f^\tau(s, s') = \chi_{[0, T]}(s'). \quad (5. 12)$$

One may consider for example the following functions

$$f_\tau(s, s') = \frac{1}{T} \sum_{n=-[T/\tau]}^{[T/\tau]} e^{i \frac{2\pi n}{T} (s-s')}. \quad (5. 13)$$

For practical purposes these are well approximated by the Gaussians (as long as $T \gg \tau$)

$$f^\tau(s, s') = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(s-s')^2}{2\tau^2}}. \quad (5. 14)$$

Thus we may define the elementary probabilities in analogy to (5. 6) as

$$p^\tau(t) = \int ds ds' \sqrt{f^\tau(t, s)} \sqrt{f^\tau(t, s')} \rho(s, s'), \quad (5. 15)$$

and construct from them the probabilities for any set $U \subset [0, T]$ as

$$p^\tau(U) = \int_U dt p^\tau(t). \quad (5. 16)$$

Up to an error of order τ this is equivalent with the probabilities obtained by coarse-graining the elementary discrete-time probabilities (5. 11)⁸.

In effect, we associate to each set U the positive operator

$$\hat{\Pi}(U) = \int_U dt \hat{R}_t \hat{R}_t^\dagger, \quad (5. 17)$$

where

$$\hat{R}_t = \int ds \sqrt{f^\tau(t, s)} \hat{C}_s \hat{P}_- \hat{H} \hat{P}_+ e^{i\hat{H}s}, \quad (5. 18)$$

is an operator that corresponds to the sum over all paths that lie within $[-\infty, 0]$ and cross through to $[0, \infty)$ within a time interval of width τ around t .

These positive operators do not yet define a POVM, because the corresponding probabilities do not add-up to unity. We have to also include the event N of no detection. The normalisation condition implies that a positive operator $\hat{\Pi}(N)$ should be defined as

$$\hat{\Pi}(N) = \hat{1} - \int_0^T dt \hat{R}_t \hat{R}_t^\dagger. \quad (5. 19)$$

The operator $\hat{\Pi}(N)$ is indeed positive, because

$$\int_0^T dt p^\tau(t) \leq \sup_{s, s' \in [0, T]} \left(\int_0^T dt \sqrt{f^\tau(t, s)} \sqrt{f^\tau(t, s')} \right) \int_0^T ds ds' \rho(s, s'). \quad (5. 20)$$

Since f^τ is a smeared delta function, the term $\int_0^T dt \sqrt{f^\tau(t, s)} \sqrt{f^\tau(t, s')}$ is maximised for $s = s'$, in which case it equals $\int_0^T dt f^\tau(t, s) = \chi_{[0, T]}(s) \leq 1$. Hence $\int_0^T dt p^\tau(t) \leq \int_0^T ds ds' \rho(s, s') = d([0, T], [0, T]) \leq 1$. Hence

$$p(N) = \text{Tr}(\hat{\rho} \hat{\Pi}(N)) = 1 - \int_0^T dt p^\tau(t) \geq 0, \quad (5. 21)$$

for all $\hat{\rho}$.

We have thus constructed a POVM for the time-of-arrival essentially by summing over all possible paths that correspond to crossing the $x = 0$ surface within a time interval of width τ ; τ was essentially introduced as a "regularisation" parameter. In general, the POVM is expected to depend strongly upon

⁸The use of the square root in (5. 15) is necessary in order to guarantee the proper dimensions of the probability density (dimensions of $[T]^{-1}$). Another way to see this is by noticing that $\sqrt{\sqrt{2\pi\tau} f^\tau(t)}$, for the Gaussian (5. 14) is a smeared characteristic function for the interval $[t - \sqrt{2\pi\tau}, t + \sqrt{2\pi\tau}]$, thus corresponding to a smeared version of (5. 11), which needs to be divided by $\sqrt{2\pi\tau}$ in order to define a probability density. Note also that if the decoherence condition holds, Eq. (5. 15) becomes, as it must, the Gaussian smearing of the probability distribution for arrival times.

its value. The key idea employed in this derivation is that time-of-arrival probabilities are not fundamentally different from the probabilities that correspond to measurements that take place at more than one moment of time. Whenever the measured quantity is continuous, it is necessary to introduce a parameter that determines the resolution of the measuring device, and it turns out that the constructed probabilities depend strongly on this parameter. Measurements of the time-of-arrival like sequential measurements of position seem to be strongly contextual, namely to depend strongly on the specific measurement device employed in their determination [12].

Our derivation relied on two assumptions. The first one is that the reduction rule may be employed consistently for the incorporation of *any information* we may obtain about a quantum system (and not only for the results of actual measurements as in section 3.2). The second assumption is that the construction of probabilities for sequential measurements may be applied in the context of time-of-arrival measurements through a generalisation of Eq. (5. 10). The key mathematical input arises from the histories description, namely the fact that it is possible to construct a sample space for the values of the time-of-arrival by considering continuous-time-histories of the system.

5.3 An explicit calculation: the free particle

We shall now compute the POVM (5. 17) explicitly for the case of a free particle. This case is particularly interesting, because it allows the comparison with a well-established result, namely the POVM constructed by Kijowski [4]. Kijowski's POVM for the time-of-arrival of a free particle assigns to any pure state ψ_0 a probability density $p(t, \psi_0)$, which is normalised to unity in the interval $(-\infty, \infty)$,

$$p(t, \psi_0) = \left| \int_0^\infty dp \left(\frac{p}{2\pi m} \right)^{1/2} e^{-ip^2 t/2M} \psi(p) \right|^2 + \left| \int_{-\infty}^0 dp \left(\frac{-p}{2\pi m} \right)^{1/2} e^{-ip^2 t/2M} \psi(p) \right|^2. \quad (5. 22)$$

To construct the POVM (5. 17) for a free particle we use Eq. (4. 33) for the decoherence functional. Since the integration in (5. 15) involves the square roots of the smeared delta-functions, which have a width of order τ , we may within an error of order $O(\tau/T)$ substitute the range of integration $\int_0^T ds \int_0^T ds' \rightarrow \int_{-\infty}^\infty ds \int_{-\infty}^\infty ds'$ and employ the Gaussian smearing functions (5. 14).

The probability density associated to (5. 17) can be written in the momentum representation as follows

$$p(t) = \frac{1}{2\pi} \int dp \int dp' \frac{p p'}{M^2} R(p, p', t) \tilde{\psi}_0(p) \tilde{\psi}_0^*(p'), \quad (5. 23)$$

where $\tilde{\psi}_0$ is the Fourier transform of ψ_0 and

$$R(p, p', t) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \sqrt{f^\tau(t, s)} \sqrt{f^\tau(t, s')} e^{-i\frac{p^2}{2M}s + i\frac{p'^2}{2M}s'} \sqrt{\frac{M}{2\pi i(s - s')}}. \quad (5. 24)$$

Changing variables to $u = \frac{1}{2}(s + s')$ and $v = s - s'$, we note that

$$\sqrt{f^\tau(t, s)} \sqrt{f^\tau(t, s')} = f_\tau(u - t) e^{-\frac{v^2}{8\tau^2}}. \quad (5. 25)$$

Within an error of order $O(\tau/T)$, the function $f_\tau(u - t)$ behaves as a delta-function when integrated over u , thus leading to

$$p(t) \simeq \frac{1}{2\pi} \int dp \int dp' \frac{pp'}{M^2} e^{-i(\frac{p^2}{2M} - \frac{p'^2}{2M})t} r\left(\frac{E_p + E_{p'}}{2}\right) \tilde{\psi}_0(p) \tilde{\psi}_0^*(p'), \quad (5. 26)$$

where $E_p = \frac{p^2}{2M}$ and

$$\begin{aligned} r(\epsilon) &= \sqrt{\frac{M}{2\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-\frac{v^2}{8\tau^2} - i\epsilon v}}{\sqrt{iv}} \\ &= \sqrt{\frac{2M\tau}{\pi}} \int_0^{\infty} dy \frac{e^{-\frac{y^2}{2}} [\cos(2\epsilon\tau y) + \sin(2\epsilon\tau y)]}{\sqrt{y}}. \end{aligned} \quad (5. 27)$$

The integral in Eq. (5. 27) can be computed explicitly in terms of modified Bessel functions; however the physically interesting information is found in specific regimes for which $r(\epsilon)$ takes a simple form. For $\epsilon\tau \ll 1$, the leading contribution to the integral is a constant, leading to

$$r(\epsilon) = \frac{\Gamma(\frac{1}{4})}{2^{3/4}} \sqrt{\frac{M\tau}{2\pi}}, \quad (5. 28)$$

which implies that the probability density $p(t)$ is proportional to $\tau^{1/2}$, when $\tau \rightarrow 0$. Hence, at the limit that the temporal resolution tends to zero the particle is never detected crossing the surface $x = 0$.

The physically interesting regime corresponds to $\epsilon\tau \gg 1$. The parameter ϵ appears in (5. 26) as the particle's energy, while τ is the temporal resolution of the measurement device. According to a common interpretation of the time-energy uncertainty principle, τ cannot be smaller than $(\Delta E)^{-1}$, where ΔE is the energy spread of the wave-functions. Hence for any wave-function with relatively small energy spread ($\Delta E/E \ll 1$), one expects that $\tau E \gg 1$. In general, it suffices that the wave-function has support only for values of energy much larger than $\frac{1}{\tau}$. The resolution τ is by assumption much smaller than t_{cl} , hence this range of energies is well defined, whenever $E t_{cl} \gg 1$. Since $t_{cl} = \frac{ML}{\bar{p}}$, where \bar{p} is the initial state's mean momentum, the condition above is equivalent

to $\bar{p}L \gg \hbar$, which is always satisfied in any macroscopic configuration for the measurement of the time-of-arrival.

At the limit $\epsilon\tau \gg 1$,

$$\int_0^\infty dy \frac{e^{-\frac{y^2}{2}} [\cos(2\epsilon\tau y) + \sin(2\epsilon\tau y)]}{\sqrt{y}} \simeq \sqrt{\frac{\pi}{\epsilon\tau}}, \quad (5. 29)$$

hence the dependence on τ drops from the probability density (5. 26)

$$p(t) \simeq \frac{1}{\pi} \int dp \int dp' \frac{pp'}{M\sqrt{\frac{1}{2}(p^2 + p'^2)}} e^{-i(\frac{p^2}{2M} - \frac{p'^2}{2M})t} \tilde{\psi}_0(p) \tilde{\psi}_0^*(p'), \quad (5. 30)$$

The POVM (5. 30) is defined for positive values of time, and for wave functions that satisfy $\hat{P}_+|\psi_0\rangle = 0$. To compare (5. 30) with Kijowski's POVM of Eq. (5. 22), which is normalised to unity by integration over for all times $t \in (-\infty, \infty)$, it is convenient to also extend the domain of the probability distribution (5. 30) to the whole real axis for time, by requiring that the extended POVM is invariant under the combined action of the parity and time-reversal transformations [4]. We employ the convention that the negative times correspond to the crossing of $x = 0$ from the right, i.e. to initial states that have support on values of position $x \in [0, \infty)$. We then construct an equal-weight convex combination of the probability distribution (5. 30) for positive t with its counterpart for negative t . We thus obtain the extension of the probability distribution (5. 30) for all real values of time and all initial wave-functions $\tilde{\psi}_0(p)$

$$p_{ext}(t) \simeq \frac{1}{2\pi} \int dp \int dp' \frac{pp'}{M\sqrt{\frac{1}{2}(p^2 + p'^2)}} e^{-i(\frac{p^2}{2M} - \frac{p'^2}{2M})t} \tilde{\psi}_0(p) \tilde{\psi}_0^*(p'). \quad (5. 31)$$

If the wave-function is sharply concentrated around the mean value \bar{p} , i.e. if $\Delta p \ll |\bar{p}|$, then $p^2 + p'^2 = 2pp' + O((\Delta p/\bar{p})^2)$ within the integration in (5. 30). The probability density (5. 31) is then identical with (5. 22). Integrating $p_{ext}(t)$ over $t \in (-\infty, \infty)$ we obtain

$$\int_{-\infty}^\infty dt p_{ext}(t) = 1 - \int dp \tilde{\psi}_0^*(-p) \tilde{\psi}_0(p). \quad (5. 32)$$

We see therefore that $p_{ext}(t)$ is normalised to unity, if the state $\tilde{\psi}_0(p)$ has support only on positive (or only on negative) values of momentum. In this case, $\hat{\Pi}(N) = 0$, i.e. all particles are eventually detected. In general, a non-zero probability $p(N)$ of non-detection is due to the fraction of particles in the statistical ensemble, which move away from the crossing surface $x = 0$.

We conclude therefore that the POVM (5. 17) leads to the same probability distributions for the time-of-arrival with Kijowski's POVM (5. 22)

for all initial wave-functions that (i) have support in values of momentum $|p| \gg \sqrt{\frac{2m}{\tau}}$ and (ii) satisfy $\Delta p \ll |\bar{p}|$. This regime includes the classical limit (e.g. wave-functions of the form (4. 34) with $\sigma_0 p \gg 1$), but also states that are not sharply localised in position, like superpositions of macroscopically distinct wave-packets, or even superpositions of states with different value of momentum—as long as Δp remains much smaller than p . Outside this regime, the POVMs (5. 17) and (5. 22) provide different predictions.

To make the points above more explicit we consider an initial Gaussian state

$$\tilde{\psi}_0(p) = (a^2/2\pi)^{1/4} \exp\left(-\frac{a^2}{4}(p - \bar{p})^2 + ipL\right) \quad (5. 33)$$

of mean position $-L$ and mean momentum \bar{p} . The momentum spreads equals to a . In leading order to $\frac{1}{a\bar{p}}$ the probability distributions for the time-of-arrival predicted by the POVMs (5. 30) and (5. 22) coincide

$$p(t) \simeq \frac{\bar{p}}{Ma\sqrt{8\pi}} e^{-\frac{2}{a^2}(L - \bar{p}t/M)^2}. \quad (5. 34)$$

The difference between the two POVMs is of order $1/(\bar{p}a)^2$ and even for a relatively large value $\frac{1}{\bar{p}a} \simeq 0.1$ as in Fig. 1 their graphs are practically indistinguishable.

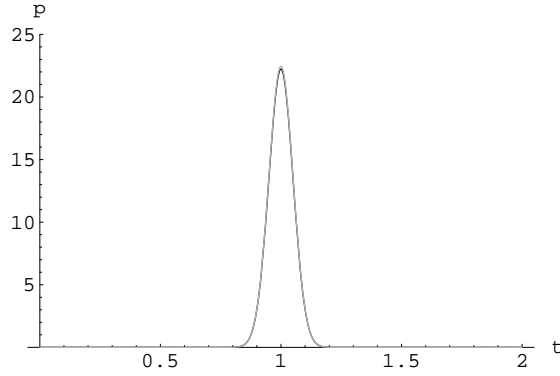


Figure 1: The probability distribution of the time-of-arrival for a Gaussian initial state (5. 33) of mean position $-L$ and mean momentum \bar{p} . It is sharply peaked around the value $t_{cl} = ML/\bar{p}$. In fact, this plot contains the probability distributions provided by both POVMs (5. 30) and (5. 22), but even for the relatively large value of $\Delta p = 0.1\bar{p}$ we employed here, the two curves almost coincide.

We next consider an initial state, which is a superposition of two Gaussians with the same mean position $-L$, but different mean momenta \bar{p}_1 and \bar{p}_2

$$\tilde{\psi}_0(p) = \left(\frac{a^2}{4\pi}\right)^{1/4} \left[e^{-\frac{a^2}{4}(p - \bar{p}_1)^2 + ipL} + e^{-\frac{a^2}{4}(p - \bar{p}_2)^2 + ipL} \right]. \quad (5. 35)$$

For $a\bar{p}_1 \gg 1$, $a\bar{p}_2 \gg 1$ and $a|\bar{p}_1 - \bar{p}_2| \gg 1$ the leading contribution to the probability distribution obtained by the POVM (5. 22) is

$$p(t) \simeq \frac{1}{Ma\sqrt{8\pi}} \left[\frac{\bar{p}_1}{2} e^{-\frac{2}{a^2}(L-\bar{p}_1 t/M)^2} + \frac{\bar{p}_2}{2} e^{-\frac{2}{a^2}(L-\bar{p}_1 t/M)^2} + \sqrt{\bar{p}_1 \bar{p}_2} e^{-\frac{2}{a^2}(L-\frac{1}{2}(\bar{p}_1+\bar{p}_2)t/M)^2} \cos\left(\frac{\bar{p}_1^2 t}{2M} - \frac{\bar{p}_2^2 t}{2M}\right) \right], \quad (5. 36)$$

and the leading contribution to the probability contribution obtained by the POVM (5. 30) is

$$p(t) \simeq \frac{1}{Ma\sqrt{8\pi}} \left[\frac{\bar{p}_1}{2} e^{-\frac{2}{a^2}(L-\bar{p}_1 t/M)^2} + \frac{\bar{p}_2}{2} e^{-\frac{2}{a^2}(L-\bar{p}_1 t/M)^2} + \frac{\bar{p}_1 \bar{p}_2}{\sqrt{\frac{1}{2}(\bar{p}_1^2 + \bar{p}_2^2)}} e^{-\frac{2}{a^2}(L-\frac{1}{2}(\bar{p}_1+\bar{p}_2)t/M)^2} \cos\left(\frac{\bar{p}_1^2 t}{2M} - \frac{\bar{p}_2^2 t}{2M}\right) \right]. \quad (5. 37)$$

We see that near the two classical values of the time of arrival that correspond to each of the two wavepackets, the two probability distributions coincide. However, for intermediate values of time, they differ. They both manifest an oscillatory behaviour there, which is characteristic of interference between the two classical values of the time-of-arrival. From Eqs. (5. 36) and (5. 37) we readily see that the oscillation amplitudes are different in this regime and their ratio is given by the quantity $\frac{1}{2} \left(\frac{\bar{p}_1}{\bar{p}_2} + \frac{\bar{p}_2}{\bar{p}_1} \right)$. When this becomes appreciably larger than unity, i.e. if the difference in mean momentum between the two Gaussians becomes comparable with the mean momenta themselves, the behaviour of the two distributions in the oscillatory region becomes substantially different.

It is important to emphasise the domain of applicability of the POVM (5. 17) is much larger than that of the free particle case (for a generalisation of Kijowski's distribution see also [32]). It can be in principle applied for systems described by arbitrary Hamiltonians. Moreover, it is constructed through a general argumentation that does not refer only to the time-of-arrival and it can be easily generalised to systems with finite-dimensional Hilbert spaces, for which there is no analogue of (5. 22). To see this one may consider equations (5. 17) and (5. 18). The only objects appearing in the definition of the POVM is the Hamiltonian (together with its propagator \hat{C}_t) and the projection operators corresponding to the two alternatives of detection. The POVM (5. 17) is therefore completely general. It can be applied for example the description of the particle being coupled to a measuring device, in which case the alternatives will correspond to projectors of the device's Hilbert space, and the Hamiltonian will contain an interaction term between particle and measurement device. It can also accommodate interactions with the environment—i.e. further terms in the Hamiltonian. Its more immediate application would be the study of tunneling

probabilities. This POVM may also refer to systems other than particles (e.g. multi-level atoms). Its derivation is only based on properties of Hilbert space operators and for this reason it can be applied to any physical context.

We chose to elaborate on the free particle case and ignore the effects of any measuring devices. The reason is that this system contains no other parameters other than the particle's mass (no couplings) and for this reason the only relevant time scale is t_{cl} . This allowed us to identify a physically relevant regime in which the time-of-arrival probabilities do not depend strongly on τ . Thus we were able to compare our result with Kijowski's POVM. This, however, cannot be expected to hold for general systems, which may involve time-scales of the same order of magnitude or smaller than τ . In the general case, the POVM (5. 17) is expected to have a strong dependence on τ even in physically relevant regimes.

5.4 The problem of contextuality

We saw that in the free particle case, there exists a regime, in which the time-of-arrival probabilities are rather insensitive to τ , but this simple behaviour cannot be expected to hold for systems with more complicated Hamiltonians that involve additional time-scales. The POVM (5. 17) will in general be strongly dependent on τ , and hence the probabilities for time will be strongly dependent upon the measurement scheme employed for their determination.

This contextuality of time-measurements in quantum theory has been emphasised by Landauer in his study of tunneling times [33] (see also a related discussion with reference to the quantum Zeno effect [10]). However, this result is not a consequence of time being a special or distinguished variable. This type of contextuality is generic in quantum theory, once we consider measurements that do not refer to a single moment of time—e.g. sequential measurements of a continuous variable. This is a necessary consequence of the formalism of quantum theory: the evolution of the quantum state involves a linear law, while probabilities are quadratic with respect to the state. Hence, it is (in general) impossible to construct a natural probability measure for the outcomes of any measurement that reveals information that pertain to a system's dynamics (sequential, time-of-arrival, continuous measurements). This problem can be seen from different angles: from the fact that the dependence on time of the quantum probabilities do not define a probability measure and hence the continuous-time limit is not well defined (as in section 5.2); from the fact that the natural measure for histories (4. 2) is non-additive; from the necessity to regularise the path integral amplitudes for the time-of-arrival in order to define probabilities; from the fact that there is interference between different alternatives for the value of the time-of-arrival. One therefore has to introduce an additional structure (external to the physical parameters of the system). The simplest such structure is the specification of the most fine-grained outcomes that can be recorded by a measurement device. In the case of observables with discrete spectrum, this

is provided naturally by the formalism. For continuous variables, however, it is not, and this brings inevitably the introduction of a scale for the fine-grained alternatives.

We cannot evade this problem by enlarging our system, including for example a quantum measuring device or an environment. The problem is a consequence of the interplay between the quantum probability rule and the unitary dynamics. It will appear in any closed system (and will be accompanied by the quantum Zeno effect). Indeed, our arguments here were very general and hold with few modifications for an arbitrary Hilbert space and with reference to the detection of any quantum event. To avoid this problem (which can take a rather extreme form [12]) we have to abandon either the probability rule or the dynamics, and neither one of these steps is easy to take.

On the other hand, the acceptance of this contextuality is very disturbing. The devices that determine the time of arrival are not different in nature from the ones that measure a particle's position. (This is reflected in the fact that the histories for the time-of-arrival are written in terms of spectral elements of the position operator.) The only distinctive character in the time-of-arrival measurements is that the "observable" quantity is the reading of the clock, which is associated to the time of detection. In position measurements, however, the fuzziness due to the finite resolution of the measurement device is relatively small, when the sampling of the measurement results are sufficiently coarse-grained. On physical grounds one would expect that coarse-graining at a scale much larger than the temporal resolution of the measurement device would give results independent of the device. Unlike the case of the free particle there is no reason to expect this for a general Hamiltonian—unless one considers the highly coarse-grained samplings around the classical equations of motion.

6 Conclusions

We have considered the problem of constructing a probability density for the time-of-arrival. Our main guideline was the fact that time appears as an external parameter in quantum theory. We relied on the histories formalism, because they allow the natural definition of probabilities about the time-of-arrival.

In our perspective, the most severe problem in the determination of the time-of-arrival probability is the fact that the quantum states do not correspond to densities with respect to time. For this reason it is very difficult to obtain the continuous-time limit in a natural way. The naive way of taking the continuous-time limit gives very bad results, as it is plagued by the quantum Zeno effect. The first alternative we tried is to employ a more strict operational interpretation of the wave packet reduction, namely that it can only be applied as a result of a system's *physical* interaction with a measurement device, and not when we obtain information about the system through inference from the lack of a detection signal. Again, the continuous-time limit was pathological

and involved the introduction of an arbitrary temporal resolution.

We then considered this problem in the light of the consistent histories approach. This suggests that the continuous-time limit should be taken at the level of amplitudes and not of probabilities, and for this reason it can be taken unambiguously. The consistent histories framework, however, is not sufficient for the definition of probabilities—there is ‘interference’ between different values of the time-of-arrival. This problem is aggravated by the presence of the quantum Zeno effect.

Nonetheless, the mathematical benefits conveyed by the histories techniques are very important and prove essential for the construction of a POVM for the time-of-arrival (working however within the operational formulation of quantum mechanics). The consideration of measurements smeared in time allows us to construct a POVM of general validity for the time-of-arrival, in analogy with POVMs for the probabilities of sequential measurements. For free particles, this POVM reduces to one obtained by Kijowski. For a general system, however, the constructed POVM also depends strongly on the resolution of the measurement device. This seems to imply that the measurement of the time-of-arrival is highly contextual within standard quantum theory.

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A The quantum Zeno effect is not robust

We provide here a simple example demonstrating the quantum Zeno effect is not robust, in the sense that even a small deviation from a projection operator in the definition (3. 6) of the operator \hat{C}_t yields to a qualitatively different behaviour.

We consider a spin system: the Hilbert space is \mathbf{C}^2 , and we choose the Hamiltonian \hat{H} and projector \hat{E} to correspond to the matrices

$$H = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{A. 1})$$

We consider the self-adjoint operator

$$\hat{V} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad (\text{A. 2})$$

from which we define the positive operator

$$e^{-V\delta t} = \begin{pmatrix} e^{-x\delta t} & 0 \\ 0 & e^{-y\delta t} \end{pmatrix} \quad (\text{A. 3})$$

This operator can be seen as a regularised expression for \hat{E}

$$\lim_{x \rightarrow \infty} \lim_{y \rightarrow 0} e^{-V\delta t} = \hat{E}. \quad (\text{A. 4})$$

We may then write a regularised version \hat{K}_t^y of the operator $\hat{C}_t = (\hat{E}e^{-i\hat{H}t/n})^n$, such that.

$$\hat{C}_t = \lim_{y \rightarrow 0} \hat{K}_t^y, \quad (\text{A. 5})$$

The operator \hat{K}_t^y reads explicitly

$$\hat{K}_t^y = \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} (e^{-i\hat{H}t/n} e^{-\hat{V}t/n})^n = \lim_{x \rightarrow \infty} e^{-iHt - Vt} \quad (\text{A. 6})$$

We easily find that

$$\hat{K}_t^y = e^{-yt} \hat{E}, \quad (\text{A. 7})$$

has the exponential fall behavior that characterises Fermi's golden rule.

When the limit $y \rightarrow 0$ is also taken, we obtain the familiar result $\hat{C}_t = \hat{E}$, which is trivially a degenerate unitary operator. However, even a small deviation from \hat{E} in the definition of \hat{C}_t leads to a different (and intuitively more physical) qualitative behaviour.